On the Theory of Bodily Tides.

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Abstract. Different theories of bodily tides assume different forms of dependence of the angular lag δ upon the tidal frequency χ. In the old theory (Gerstenkorn 1955, MacDonald 1964, Kaula 1964) the geometric lag angle is assumed constant (i.e., $\delta \sim \chi^0$), while the new theory (Singer 1968; Mignard 1979, 1980) postulates constancy of the time lag Δt (which is equivalent to saying that $\delta \sim \chi^1$).

Each particular functional form of $\delta(\chi)$ unambiguously determines the form of the frequency dependence of the tidal quality factor, $Q(\chi)$, and vice versa. Through the past 20 years, several teams of geophysicists have undertaken a large volume of experimental research of attenuation at low frequencies. This research, carried out both for mineral samples in the lab and for vast terrestrial basins, has led to a complete reconsideration of the shape of $Q(\chi)$. While in late 70s - early 80s it was universally accepted that at low frequencies the quality factor scales as inverse frequency, by now it is firmly established that $Q \sim \chi^{\alpha}$, where the positive fractional power α varies, for different minerals, from 0.2 through 0.4 (leaning toward 0.2 for partial melts) – see the paper by Efroimsky (2006) and references therein. That paper also addresses some technical difficulties emerging in the conventional theory of land tides, and offers a possible way of their circumvention – a new model that is applicable both for high inclinations and high eccentricities (contrary to the Kaula expansion which converges only for $i \neq \pi/2$ and $e < 0.6627434$). Here we employ this new model to explore the long-term evolution of Phobos and to provide a more exact estimate for the time it needs to fall on Mars. This work is a pilot paper that anticipates a more comprehensive study in preparation (Efroimsky & Lainey 2007).

Keywords: Land tides, body tides, bodily tides, tidal forces, tidal torques, tidal interactions, Mars, Phobos, Deimos.
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NOTATIONS AND ASSUMPTIONS

If a satellite is located at a planetocentric position $\vec{\mathbf{r}}$, it generates a tidal bulge that either advances or retards, depending on the interrelation between the planetary spin rate ω_p and the tangential part of satellite's velocity $\vec{\mathbf{v}}$ divided by $r \equiv |\vec{\mathbf{r}}|$. It is convenient to imagine that the bulge emerges beneath a fictitious satellite located at

$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}} + \vec{f} \quad , \tag{1}$$

where the position lag $\vec{f}$ is given by

$$\vec{f} = \Delta t \; (\; \vec{\omega}_p \times \vec{\mathbf{r}} \; - \; \vec{\mathbf{v}} \;) \quad , \tag{2}$$

Δt being the (assumed to be small) time lag between the real and fictitious tide-generating satellites. We shall also introduce the angular lag as

$$\delta \; \equiv \; \frac{|\vec{f}\,|}{r} \; = \; \frac{\Delta t}{r} \, |\vec{\omega}_p \times \vec{\mathbf{r}} \; - \; \vec{\mathbf{v}}| \quad . \tag{3}$$

For a circular orbit, δ is the absolute value of the angle subtended at the planet's centre between the satellite and the tidal bulge. At nonzero eccentricities, our δ differs from the subtended angle. We have deliberately arranged for this difference, to ensure that our δ reflects not only the "horizontal" but also the "vertical" lagging, i.e., the lag in the position but also the lag in the height of the bulge (Murray & Dermott 1999, pp. 170 - 171). Our δ defined through (3) turns out to be equal to $\sqrt{(\text{horizontal lag})^2 + (\text{vertical lag})^2}$. This, together with the interconnection between the lag angle and the quality factor, ensures that our model indeed includes both lags – that in the position and that in the height of the tide.

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One of the advantages of this definition of δ is that, in distinction from the Gerstenkorn-Kaula-MacDonald theory, our model will permit for studying tidal dissipation in spin-orbit-resonance setting, even though the subtended angle in such settings averages to zero. This will make our model applicable to tides generated by the planet on the satellite, provided we take into account only the optical, not the physical libration.[1]

THE TIDAL FREQUENCY χ AND THE QUALITY FACTOR Q

What is the tidal frequency?

For an equatorial circular orbit, the satellite velocity relative to the surface is

$$|\vec{\omega}_p \times \vec{\mathbf{r}} - \vec{\mathbf{v}}| = r\,|\omega_p - n| \quad , \tag{4}$$

the main tidal frequency is

$$\chi = 2\,|\omega_p - n| \quad , \tag{5}$$

and the angular lag is

$$\delta \equiv \frac{\Delta t}{r}\,|\vec{\omega}_p \times \vec{\mathbf{r}} - \vec{\mathbf{v}}| = \frac{\Delta t}{2}\,\chi \quad . \tag{6}$$

How to extend this formalism to the generic case when neither the inclination i nor the eccentricity e is small? Specifically, how can one define the tidal frequency (or, perhaps, the principal tidal frequency and the higher frequencies) when the satellite may appear over a different point on the surface after each revolution, thus making the entire notion of flexure cycle hard to define?

Generalisation of the above construction was offered already by Darwin (1908) and Jeffreys (1961), and it was greatly advanced by Kaula (1964). These works' starting point was that each elementary volume of the planet is subject to a tide-raising potential, which in general is not periodic but can always be expanded into a sum of periodic terms. All three authors then asserted that the overall dissipation inside the planet may be represented as a sum of attenuation rates corresponding to each periodic disturbance. This approach extends (and strains) the linearity approximation, approximation according to which the tidal perturbations of the potential yield linear response of the shape and linear variations of the stress.

A serious objection to this method was put forward by Goldreich (1963) who wrote that "*a tide of small amplitude will have a phase lag which increases when its peak is reinforcing the peak of the tide of the major amplitude.*" Thence Goldreich derived a conclusion that we *"use the language of linear tidal theory, but we must keep in mind that our numbers are really only parametric fits to a non-linear problem."*

To this, it should be added that whenever we need to explore a tidal capture (which happens at high eccentricities), the Kaula expansion becomes useless, because it converges only for $e < 0.6627434$. Similarly, it becomes inapplicable when one needs to explore if the tidal forces are capable of transforming a retrograde orbit into a prograde one. In case such events ever took place, they implied a transition through a polar orbit – situation that cannot be tackled by the Kaula machinery, because his expansion will diverge at $i = 90$.

An alternative approach

The method developed in Efroimsky (2006) is an attempt to circumvent the afore mentioned difficulties of the traditional, linear approach. Within that method, there exist neither a main frequency nor higher-order ones, but just one instantaneous tidal frequency χ, which depends on the instantaneous position of the satellite. This frequency is defined as the doubled velocity of the satellite relative to the planet's surface, divided by the satellite's planetocentric

[1] In most situations, the amplitude of the physical librations is two orders, or at least an order less than the effect from the optical libration.

radius r. Expressed via the satellite's inertial velocity $\vec{\mathbf{v}}$ and planetocentric position $\vec{\mathbf{r}}$ and the planet's spin rate $\vec{\omega}_p$, this frequency reads:

$$\chi \;=\; \frac{2}{r}\;|\,\vec{\omega}_p\times\vec{\mathbf{r}}\;-\;\vec{\mathbf{v}}\,|\quad, \tag{7}$$

Only this instantaneous frequency is used to determine the offset of an equilibrium tidal bulge from the direction to the satellite. When expressed through the satellite's orbital parameters, it naturally turns out to be a function of the true anomaly of the satellite:[2]

$$\chi = 2\,\sqrt{(\dot{\nu}-\omega_p\cos i)^2+\omega_p^2\sin^2 i\,\cos^2(\omega+\nu)+\mathrm{v}_r^2\,r^{-2}}$$

$$=\;2\;\sqrt{\left[n\,\frac{(1+e\cos\nu)^2}{(1-e^2)^{3/2}}\;-\;\omega_p\cos i\right]^2\;+\;\omega_p^2\,\sin^2 i\;\cos^2(\omega+\nu)\;+\;n^2\,e^2\,\frac{(1+e\cos\nu)^2}{(1-e^2)^3}\;\sin^2\nu}\quad, \tag{8}$$

Be mindful that χ depends not only upon the horizontal velocity of the subsatellite point but also upon the vertical component of the satellite's velocity, v_r. This term in (8) is of order e^2, it reflects the afore mentioned fact that tidal dissipation should include a contribution from both the position and height lags of the bulge.

Referring the reader to Efroimsky (2006) for technical details, here we shall mention two principal advantages of the method. First, when χ is defined through (7), the interrelation (6) between δ and χ extends to arbitrary eccentricities and inclinations. Second, definition (7) satisfies the dissipation equation

$$\Delta E_{_{cycle}}(\chi)\;=\;-\;2\,\pi\,\frac{E_{_{peak}}}{Q(\chi)}\quad, \tag{9}$$

$Q(\chi)$ being the effective quality factor of the entire tidal bulge (not of a particular harmonic χ, as in the linear case).

THE QUALITY FACTOR Q AND THE GEOMETRIC LAG ANGLE δ

The basic formula

The tidal quality factor is interconnected with the phase lag ε and the angular lag δ via

$$Q^{-1}\;=\;\tan\varepsilon\;=\;\tan 2\delta \tag{10}$$

or, for small lags,

$$Q^{-1}\;\approx\;\varepsilon\;=\;2\,\delta\quad. \tag{11}$$

Derivation of relation (10) can be found[3], for example, in Goldreich (1963) on page 260, or in Kaula (1964) on page 673. A much simplified derivation, serving solely illustrative purposes, can be found in the Appendix to Efroimsky (2006). Unfortunately, some authors mistakenly say that Q^{-1} is equal simply to $\tan\delta$, with the factor of two omitted.[4]

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[2] The idea of introducing a position-dependent instantaneous frequency is borrowed from quantum mechanics where it is used to build the approximate WKB solution to the Schrödinger equation. This analogy, though, should not be taken too literally, because in quantum mechanics this method is used to solve a linear equation, while in Efroimsky (2006) it was employed to cope with nonlinearity.

[3] Actually, the geometric lag angle used in these references is simply the subtended angle and therefore differs from our *delta*, as mentioned in the first section of this paper. However, we would insist that in equations (10) and (11) one should have our δ defined through (3) rather than the subtended angle. This is necessary to reflect the fact that Q describes both the dissipation associated with the position lag of the bulge and the dissipation associated with its height lag.

[4] This oversight was made, for example, by Rainey & Aharonson (2006), who therefore obtained for Q a value about twice larger than the actual one. In the paper by Bills et al (2005), one letter, γ, is used to denote two different angles. Prior to formula (24), γ signifies the *geometric* lag (in our notations, δ). Further, in formulae (24) and (25), the authors use γ to denote the *phase* lag (in our notations, ε, which is equal to 2δ). With this crucial caveat, Bills' equation $Q=1/\tan\gamma$ is correct. Fortunately, this oversight in notations has not prevented Bills et al (2005) from arriving to a realistic value of the Martian Q factor, 85.58 ± 0.37. A later study by Lainey et al (2006) has given a comparable value of 79.91 ± 0.69.

While Gerstenkorn (1955), MacDonald (1964) and Kaula (1964) assumed the subtended angle to be a fixed constant, Singer (1968) suggested that the subtended angle should be proportional to the principal frequency χ of the tide. This was equivalent to setting Δt constant in (2 - 3). A comprehensive development of Singer's idea was performed by Mignard (1979, 1980) who avoided using the lag angle and operated only with the position and time lags. Later, Singer's assumption of a constant Δt was accepted by Touma & Wisdom (1994) and Peale & Lee (2000).

From (11), it can be seen that the Gerstenkorn-MacDonald-Kaula theory implies $Q \sim \chi^0$, while the Singer-Mignard theory yields $Q \sim \chi^{-1}$. Unfortunately, neither of these options is in agreement with the geophysical data.

Attenuation in minerals at low frequencies.

Back in 60s and 70s, it was widely believed among seismologists that at low frequencies the quality factor should scale as inverse frequency. It is at about that time when this perception proliferated into the astronomical community, where it was received most warmly. The reason for the popularity of this concept among astronomers is that this is the only model for which the linear expansion of the tide gives a system of bulges displaced from the direction to the satellite by the same angle. All other frequency dependencies of Q lead to superposition of bulges corresponding to the separate frequencies in the expansion, each bulge being displaced by its own unique angle. This is the reason why laying the convenient dependence $Q \sim 1/\chi$ to rest will be a wrench for any astronomer. Nevertheless, the progress achieved in the low-frequency seismology over the past twenty years tells us with all certainty that this mathematically convenient dependence has nothing to do with reality. Departure from this scaling law was slow, due to the technical difficulties with performing measurements at too long time scales. While a review of the pertinent literature can be found in Efroimsky (2006), here we shall present the final outcome.

At frequencies lower than $\sim 1\ \mathrm{yr}^{-1}$ attenuation in mantle is defined by viscosity, so that the quality factor is, for all minerals, well approximated with $\eta\chi/M$, where η and M are the sheer viscosity and the sheer elastic modulus of the mineral. Although both the sheer and stretch viscosity coefficients, as well as the elastic moduli, greatly vary for different materials and are sensitive to the temperature, the overall quality factor of the mantle behaves as

$$Q \sim \chi \quad . \tag{12}$$

This immediately entails, through (11), that for near-synchronous satellites we must accept

$$\delta_{near-synchr} \sim \chi^{-1} \quad , \tag{13}$$

because for them $\chi \equiv 2|\omega_p - v_T/r| < 1\ \mathrm{yr}^{-1}$.

As most moons in the Solar system are far from synchronicity with their primaries, then for practical purposes we need to know the behaviour of Q at higher frequencies χ. That this behaviour has the form of

$$Q \sim \chi^{\alpha} \quad , \quad \text{with } \alpha \text{ ranging from } 0.2 \text{ through } 0.4 \ , \tag{14}$$

has been long-established. The dependence reliably holds for all rocks within a remarkably broad band of frequencies: from $1\ \mathrm{yr}^{-1}$ all the way up to several MHz. (Karato & Spetzler 1990) Hence, for most satellites,

$$\delta \sim \chi^{-\alpha} \quad , \quad \alpha = 0.2 - 0.4 \quad . \tag{15}$$

It should be added that for partial melts the values of α typically lean toward 0.2.

Although the tidal Q differs from the seismic one, both depend upon the frequency in the same way, for this dependence is determined by the same physical mechanisms. This pertains also to the temperature dependence, which for some fundamental reason combines into one function with the frequency dependence. The theory underlying the scaling laws (12) and (14) is set out in Karato (2007).

FORMULAE

The tidal potential perturbation acting on the tide-raising satellite is[5]

$$U(\delta_1) \;=\; \frac{A_2}{r_f^5\, r^5}\,\left(3\,\left(\vec{\mathbf{r}}_f\cdot\vec{\mathbf{r}}\right)^2 - \vec{\mathbf{r}}_f^{\,2}\,\vec{\mathbf{r}}^{\,2}\right) + \frac{A_3}{r_f^7\, r^7}\,\left(5\,\left(\vec{\mathbf{r}}_f\cdot\vec{\mathbf{r}}\right)^2 - 3\,\vec{\mathbf{r}}_f^{\,2}\,\vec{\mathbf{r}}^{\,2}\right) + \;\ldots \tag{16}$$

where $r \equiv |\vec{\mathbf{r}}|$ and $r_f \equiv |\vec{\mathbf{r}}_f|$, while the constants are given by

$$A_2 \;\equiv\; \frac{k_2\,G\,m\,R^5}{2}\quad,\quad A_3 \;\equiv\; \frac{k_3\,G\,m\,R^7}{2}\quad,\quad \ldots \tag{17}$$

The ensuing formula for the tidal force will read:[6]

$$\vec{\mathrm{F}} \;=\; -\,\frac{3\,k_2\,G\,m^2\,R^5}{r^{10}}\,\left[\vec{\mathbf{r}}\,r^2 - \vec{f}\,r^2 - 2\,\vec{\mathbf{r}}\,\left(\vec{\mathbf{r}}\cdot\vec{f}\right)\right] + O(k_3 G m^2 R^7/r^9)\quad. \tag{18}$$

The interconnection between the position, time, and angular lags,

$$\delta \;\equiv\; \frac{|\vec{f}|}{r} \;=\; \Delta t\;\frac{1}{r}\,|\,\vec{\omega}_p\times\vec{\mathbf{r}} - \vec{\mathbf{v}}\,| \;=\; \frac{\Delta t}{2}\,\chi\quad, \tag{19}$$

can be equivalently rewritten as:

$$\vec{f} \,=\, \hat{\mathbf{f}}\,r\,\delta \;=\; r\,\frac{\Delta t}{2}\,\chi\,\hat{\mathbf{f}}\quad, \tag{20}$$

where

$$\hat{\mathbf{f}} \,=\, \frac{\vec{\omega}_p\times\vec{\mathbf{r}} - \vec{\mathbf{v}}}{|\,\vec{\omega}_p\times\vec{\mathbf{r}} - \vec{\mathbf{v}}\,|} \tag{21}$$

is the unit vector pointing in the lag direction, and δ is a known function (14) of the main frequency χ. Since $\delta\sim\chi^{-\alpha}$ then (19) necessitates for the time lag:

$$\frac{\Delta t}{2} \;=\; \mathrm{E}^{\,\alpha}\,\chi^{-(\alpha+1)}\quad, \tag{22}$$

and for the position lag:

$$\vec{f} \;=\; r\,\delta\,\hat{\mathbf{f}} \;=\; r\,\mathrm{E}^{\,\alpha}\,\chi^{-\alpha}\,\hat{\mathbf{f}}\quad, \tag{23}$$

E being the planet's integral parameter[7] of dimension s^{-1}, while χ is a known function (8) of the orbital variables. Putting everything together, we arrive at

$$\vec{f} \;=\; \left(\frac{\mathrm{E}}{\chi}\right)^{\alpha}\,a\,\frac{1-e^2}{1+e\,\cos\nu}\;\frac{\vec{\omega}_p\times\vec{\mathbf{r}} - \vec{\mathbf{v}}}{|\,\vec{\omega}_p\times\vec{\mathbf{r}} - \vec{\mathbf{v}}\,|} \tag{24}$$

where

$$\chi \,\equiv\, 2\,\sqrt{\left[n\,\frac{(1+e\cos\nu)^2}{(1-e^2)^{3/2}} - \omega_p\,\cos i\right]^2 + \omega_p^2\,\sin^2 i\,\cos^2(\omega+\nu) + n^2\,e^2\,\frac{(1+e\cos\nu)^2}{(1-e^2)^3}\,\sin^2\nu}\quad. \tag{25}$$

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[5] This potential is a function of the subtended angle $\delta_1 = arccos\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}_f}{|\vec{\mathbf{r}}|\,|\vec{\mathbf{r}}_f|}$. Our lag δ is, generally, different from δ_1, and coincides with it only in the limit of vanishing e.

[6] To (16) we should add the potential due to the tidal deformation of the satellite by the planet. That input contributes primarily to the radial component of the tidal force exerted on the moon, and entails a decrease in eccentricity (MacDonald 1964). Here we omit this term, because our goal is to clarify the frequency dependence of the lag.

[7] This parameter is defined through $Q = \mathrm{E}^{\,\alpha}\,\chi^{\alpha}$, so that $\mathrm{E}^{\,\alpha}$ is simply the dimensional factor emerging in the relation $Q\sim\chi^{\alpha}$.

The time lag is, according to (22):

$$\Delta t = \left(\frac{\mathrm{E}}{2}\right)^{\alpha} \left(\left[n\frac{(1+e\cos v)^2}{(1-e^2)^{3/2}} - \omega_p \cos i \right]^2 + \omega_p^2 \sin^2 i \cos^2(\omega + v) + n^2 e^2 \frac{(1+e\cos v)^2}{(1-e^2)^3} \sin^2 v \right)^{-(\alpha+1)/2} . \qquad (26)$$

THE EXAMPLE OF PHOBOS' FALL TO MARS

The afore presented formalism was employed to calculate the tidal evolution of Phobos until its fall on the Martian surface. The goal of the current work is to point out the qualitative differences between the new model of tides and the ones used in the past. So we neglected the fact that Phobos is close to its Roche limit, and may be destroyed by tides prior to its fall. We also restricted the dynamical interactions between Phobos and Mars to a two-body problem amended only by the tides raised by Phobos on Mars. We also neglected the other perturbations, like the Martian non-sphericity and precession, or the pull exerted upon Phobos by the Sun, the planets, and Deimos.

To integrate Phobos' orbital motion around Mars, the planetary equation in the Euler-Gauss form were employed:[8]

$$\left\{ \begin{aligned} \frac{da}{dt} &= 2a^2[Re\sin v + S(1+e\cos v)]\mathrm{G}^{-1} \\ \frac{de}{dt} &= p[R\sin v + S(\cos v + \cos E)]\mathrm{G}^{-1} \\ \frac{di}{dt} &= rW\cos(\omega+v)\mathrm{G}^{-1} \\ \frac{d\Omega}{dt} &= rW\sin(\omega+v)/(\mathrm{G}\sin i) \\ \frac{d\omega}{dt} &= \left(-pR\cos v + (r+p)S\sin v - \mathrm{G}\, e\cos i \frac{d\Omega}{dt}\right)/\mathrm{G}\, e \\ \frac{dM}{dt} &= n - \frac{1}{\sqrt{\mu a}}\left[2rR + \mathrm{G}\left(\frac{d\omega}{dt} + \cos i\frac{d\Omega}{dt}\right)\right] \end{aligned} \right. \qquad (27)$$

where $(a, e, i, \Omega, \omega, M)$ are the standard elliptic elements, v is the true anomaly, E is the eccentric anomaly, $\mu = G(M_0 + m)$ is the reduced mas, M_0 and m are the masses of Mars and Phobos, $\mathrm{G} = \sqrt{\mu p}$ and $p \equiv a(1-e^2)$ are the ellipse parameters, and $\vec{F} = (R, S, W)$ is the perturbative force expanded in the comoving frame.

At each step of our integration, the current value of the tidal force $\vec{F}$ was calculated from equation (18). The initial values of the orbital elements of Phobos, and the values of all the other physical parameters were borrowed from Lainey et al. (2006). These included an estimate of 0.6345 min for the present-day value of the time lag Δt. Our numerical technique was based on the RA15 integrator developed by Everhart (1985) and known for its speed and accuracy. In the cause of integrations, a variable step size with an initial value of 0.025 day was used. We estimated the numerical accuracy of our simulations to be of order one kilometer.

We performed two different computations, the first one implementing the Singer-Mignard tidal theory with a constant Δt, and the second one using the new model described above. In the latter integration, we first computed the instantaneous frequency from equation (25), and then deduced Δt from equation (22). To be sure that, despite the different tidal models, both simulations started with the same initial conditions, we fitted the integral parameter E emerging in equation (22). To that end, we compared both simulations over a span of 1000 years. The right value for E was found to be $\mathrm{E} = 10.68 \times 10^{-7}$ day^{-1}.

Figure 1 presents the evolution of Phobos' semi-major axis from its current value all way till the satellite crashes on Mars, having descended about 6000 km. It can be seen that within the Singer-Mignard model Phobos falls on Mars 15 Myr sooner than within our model. This difference emerges because within the Singer-Mignard formalism Δt stays unchanged through the descent, while in the new model it is oscillating, its average value gradually decreasing as shown on Figure 2.

[8] The advantage of this method, as compared to direct integration in a Cartesian frame, lies in the fact that the six planetary equations (in whatever form – Lagrange, or Delaunay, or Euler-Gauss –) deal solely with perturbation and having the unperturbed Keplerian part already integrated *a priori*. This trivial circumstance permits for the use of a smaller constrain in the variable-step-size criterium, thus saving computing time.

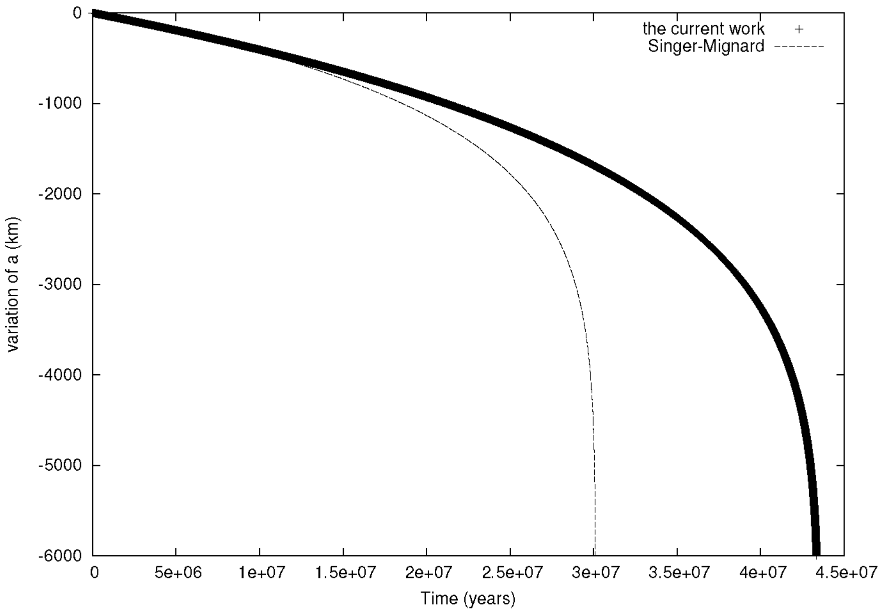

FIGURE 1. Evolution of Phobos' semi-major axis, as predicted by the Signer-Mignard's model (the thin line) and by the new model used in the current paper (the thick line).

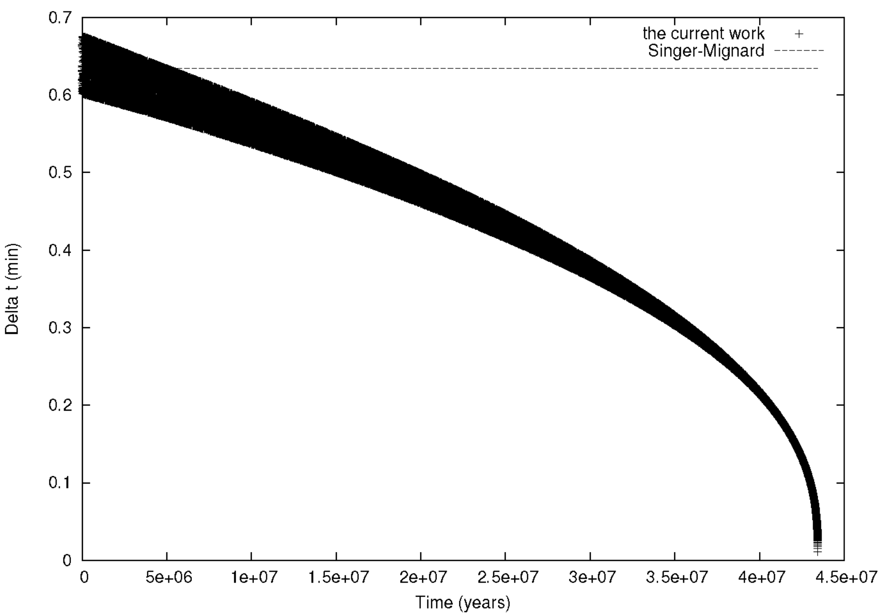

FIGURE 2. Evolution of Δt in the Signer-Mignard's model (where it is assumed constant – see the thin horizontal line) and in the new model used in the current paper (the thick line depicts the fluctuation of Δt about a gradually decreasing average).

CONCLUSIONS

Using the realistic tidal-frequency dependencies for the time and angular lags, along with the recently updated values of the Martian parameters, we explored the future history of Phobos, taking into account only the tides raised by Phobos on Mars, but not those produced by Mars on Phobos. According to our computations, Phobos will fall on Mars in 45 Myr from now. This estimate is 50 percent longer than it the one ensuing from the tidal models used in the past. This demonstrates that the currently accepted time scales of dynamical evolution, deduced from old tidal models, should be reexamined using the actual frequency dependence of the lags. The authors are presently working in this direction (Efroimsky & Lainey 2007).

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